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On the inertial mass of a vortex in high-T_c superconductors: closed form results

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Abstract. Closed form results for the vortex inertial mass due to the variation of the magnitude of the superconducting order parameter are presented for an s-wave superconductor. The evaluation of certain matrix elements which use approximate wavefunction solutions of the Bogoliubov–de Gennes equations are completed. The results exhibit the dependences upon microscopic parameters, thus improving upon previous numerical estimates. The analytic expressions more fully characterize the inertial mass and the core polarizability and are suitable for incorporation in the vortex mobility at absolute zero. The unscreened vortex-core mass is sized by using simple BCS relations.

Recently an examination of the vortex inertial mass for an s-wave superconductor has been recounted [1]. This study was based upon the Bogoliubov–de Gennes equations and examined only the absolute zero temperature case; it repeated much of the results of an earlier paper [2]. One idea was to perform a microscopic calculation of the mass, avoiding a Ginzburg–Landau (GL) treatment. However, as this paper illustrates, that discussion is incomplete in both the analytic results and in the consideration of contributing mechanisms and related literature.

Generally an inertial mass calculation is more useful when the dependence on various parameters is apparent. In particular, to more fully utilize the results of a microscopic calculation, one would like to exhibit the dependences on parameters such as the Fermi wavenumber k_F, interlayer spacing d, coherence length ξ, and zero-temperature gap amplitude Δ_0. The numerical results of [1] which finish the calculations tend to conceal such dependences. After some background remarks on the vortex mass, this paper presents closed form results first for the unscreened core contribution m_0^* to the vortex mass. Then, analytic results pertinent to the core dielectric constant ϵ_{core} are described. Specifically, one of the definite integrals in ϵ_{core} is easily evaluated and the convergence of the remaining integral is discussed with the aid of an asymptotic estimate. An analytic approximation for the core polarizability is also presented. As the vortex mass of [1, 2] is m_0^* divided by the core dielectric constant, much progress can be made in developing closed form results.

[1] used the vortex core bound-state wavefunctions of [3] and this practice is continued here. Comparison of these analytic forms with the numerical results of [4, 5] shows these to be valuable approximations.

Estimation of the vortex mass is of importance in describing dynamic vortex phenomena including radiofrequency (rf) response [6, 7] and quantum tunnelling [8]. (If it were possible to attain the necessary conditions of low temperature and very weak pinning, quantum tunnelling of vortices may occur.) A suitable function for describing vortex response is the complex-valued dynamic mobility [7, 9–11]. The mobility can simultaneously include the effects of inertia, pinning, flux flow, and flux creep. Knowledge of the vortex mass is desirable in determining whether to include an inertial term in the equation of motion or dynamic mobility. The vortex mobility can be written in the limiting case when the viscous drag force vanishes, where inertial effects should be pronounced [10].

The vortex mobility $\tilde{\mu}_v$ [9, 11] enters the general relation between velocity $\boldsymbol{v}$ and driving force $\boldsymbol{f}$ as $\boldsymbol{v} = \tilde{\mu}_v \boldsymbol{f}$. The driving force in the vortex equation of motion may be, for instance, the Lorentz force or thermal force. As an example of the dynamic mobility, for simple harmonic vortex motion, $\tilde{\mu}_v(\omega, B, T) = (-i\omega\mu + \eta + i\kappa_p/\omega)^{-1}$, where $\mu(T)$ is the vortex mass per unit length, $\eta(B, T)$ is the viscous drag coefficient, and $\kappa_p(B, T)$ is the pinning force constant. Here ω is the angular frequency, B the magnetic induction, and T the temperature. In terms of characteristic lengths, this expression can be written as $\tilde{\mu}_v(\omega, B, T) = \eta^{-1}(1 - i\mu\omega/\eta + i\delta_f^2\lambda_C^{-2}/2)^{-1}$, where $\lambda_C^2 = B\phi_0/\mu_0\kappa_p$ is the square of the Campbell (pinning) penetration depth and $\delta_f^2 = 2B\phi_0/\mu_0\eta\omega$ is the square of the flux-flow skin depth. (The flux quantum is $\phi_0 = hc/2e$.) From the mobility, the complex-valued resistivity associated with vortex motion follows as $\tilde{\rho}_v = B\phi_0\tilde{\mu}_v$.

More generally, in the presence of tensor forces acting on vortices, or for anisotropic type-II superconductors, the mobility needs to be taken as a tensor [11, 12]. In this situation, the mobility would include a viscosity tensor, a tensor set of pinning constants, and off-diagonal contributions from the Hall force. When a tensor vortex effective mass is required, as perhaps for a layered superconductor, the expression for the mobility is further complicated.

The inertial mass can arise from variation of the amplitude of the order parameter as the vortex moves through a type-II superconductor, giving a 'core' contribution, or from additional electromagnetic energy generated by the motion. The resulting inertial mass often varies directly with a characteristic critical magnetic field and inversely with the square of a characteristic speed. For a continuous type-II superconductor with Abrikosov vortex the core contribution is [13]

$$\mu_{\text{core}} = \frac{3}{8\pi}\frac{\phi_0}{v_F^2}\frac{H_{c1}}{\ln\kappa} \tag{1}$$

while the electromagnetic contribution is [12, 13]

$$\mu_{\text{em}} = \frac{\phi_0}{16\pi c^2}H_{c2} \tag{2}$$

and the rest mass is

$$\mu_0 = \frac{\phi_0}{4\pi c^2}H_{c1}. \tag{3}$$

Here v_F is the Fermi velocity, and H_{c1} and H_{c2} are the lower and upper critical fields, respectively; $\kappa = \lambda/\xi$ with λ the penetration depth is the GL parameter. For a Josephson vortex in a single Josephson junction, the mass [14] is

$$\mu_{\text{LS}} = \frac{\phi_0}{4\pi \bar{c}^2}H_{c1J} \tag{4}$$

where $\bar{c}$ is the speed of light in the insulating layer and H_{c1J} is the junction lower critical field. In equations (1)–(4) $\phi_0 H_{c.}/8\pi$ gives a characteristic magnetic energy per unit length. Due to the relatively small Fermi velocity and large critical fields H_c and H_{c2} the vortex mass can be appreciably larger in high-T_c materials compared to 'conventional' type-II superconductors [12].

In [15] an elastic mechanism leading to the inertial mass of a flux line in a type-II superconductor was introduced. In [16] were discussed some inadequacies of the derivation and the result for the mass per unit length μ_{d}. A new identity was deduced which is suitable for evaluating μ_{d} for arbitrary quasi-particle fractions when the ionic displacement is non-rotational. This result avoids approximations in the original model [15] and clarifies the temperature dependence of the fluxon mass.

The possible experimental observation of the local strain field and of other mechanisms which may be useful in studying the temperature dependence of the vortex inertial mass has been discussed for example in [10, 16, 17]. In [17] some difficulties with a proposal to define a vortex mass from the lattice elastic energy were discussed.

In [18] nonlinear vortex dynamics in an ultraclean type-II superconductor were examined. Theory for weakly nonlinear and either weakly or strongly dispersive wave propagation in a vortex lattice has been developed. It is found that integrable nonlinear wave equations govern the electrodynamics. The vortex mass per unit length plays an important role in these derivations. When there is negligible pinning and drag, the Korteweg–de Vries (KdV) and cylindrical KdV (CKdV) equations and their two-dimensional generalizations can be obtained in various geometries. When the Hall force dominates, the dispersion relation of the linearized problem is modified, leading to the nonlinear Schrödinger equation (NLS) for the complex amplitude. The detection of such KdV or NLS solitons could provide a means of studying the Hall coefficient and vortex mass at very low temperature.

1. Evaluation of unscreened core contribution

The unscreened core contribution m_0^* to the vortex mass of [1] follows from a momentum–momentum correlation function. Then, $m_0^* = 4|g_x|^2/\epsilon_{1/2}$ where $g_x = (\hbar/\mathrm{i})\int \mathrm{d}\boldsymbol{r}\, v_{1/2}(\boldsymbol{r})\partial u_{1/2}(\boldsymbol{r})/\partial x$ and $\epsilon_{1/2}$ is the energy of the first excited core state. In g_x, the momentum operator $p_x = (\hbar/\mathrm{i})[\cos\phi\,\partial/\partial r - (\sin\phi/r)\partial/\partial\phi]$. The radial dependence of the core wavefunctions is given by $f^{\pm}_{1/2}(r) = A_{1/2} J_{\substack{1\\0}}(k_{\mathrm{F}} r)\,\mathrm{e}^{-r/2\xi}$. (Here J_n is the Bessel function of order n and $A_{1/2}$ is the normalization constant.)

Since $J_0' = -J_1$ and letting $p \equiv k_{\mathrm{F}}\xi$, it follows that the matrix element g_x [1] can be written as

$$g_x = -\frac{\hbar}{2\xi}\frac{\int_0^\infty x\,\mathrm{d}x\,\mathrm{e}^{-x} J_1(px)[J_0(px)/2 + pJ_1(px)]}{\int_0^\infty x\,\mathrm{d}x\,\mathrm{e}^{-x}[J_0^2(px) + J_1^2(px)]}. \tag{5}$$

Evaluating the integrals in terms of the complete elliptic integral $\boldsymbol{E}(k) = E(\pi/2, k)$ and hypergeometric function ${}_2F_1$ [19] gives

$$g_x = -\frac{\hbar p}{4\xi}\frac{[{}_2F_1(3/2, 3/2; 2; -4p^2) + 3p^2\,{}_2F_1(3/2, 5/2; 3; -4p^2)]}{[(2/\pi)(1+4p^2)^{-1}\boldsymbol{E}(-4p^2) + (3/2)p^2\,{}_2F_1(3/2, 5/2; 3; -4p^2)]}. \tag{6}$$

It is recalled that $\boldsymbol{E}(k) = (\pi/2)\,{}_2F_1(-1/2, 1/2; 1; k^2)$.

For $\xi = 15$ Å, $k_{\mathrm{F}}^{-1} = 3.36$ Å so that $p = 4.47$ and the argument $-4p^2 = -79.9$, $\boldsymbol{E}(-4p^2) = 9.16776$, ${}_2F_1(3/2, 3/2; 2; -4p^2) = 0.004\,556\,4632$, and ${}_2F_1(3/2, 5/2, 3, -4p^2) = 0.0022553689$, giving $g_x = -1.12\hbar/\xi$. These numbers are suitable for the high-T_{c} cuprates and there is agreement with the numerical results of [1]. One can also note that the number $1.12 \simeq p/4$ and the remaining ratio in g_x is near unity for a range of p values. For large negative arguments of the hypergeometric function, as here, one can usefully employ the transformations ${}_2F_1(a, b; c; x) = (1-x)^{-a}\,{}_2F_1(a, c-b; c; x/(x-1)) = (1-x)^{-b}\,{}_2F_1(b, c-a; c; x/(x-1))$ in the series evaluation.

In summary, the unscreened core contribution to the vortex mass is given by

$$m_0^* = \frac{\hbar^2 k_F^2}{4\epsilon_{1/2}} \left[\frac{{}_2F_1(3/2,3/2;2;-4p^2) + 3p^2 {}_2F_1(3/2,5/2;3;-4p^2)}{(2/\pi)(1+4p^2)^{-1} \boldsymbol{E}(-4p^2) + (3/2)p^2 {}_2F_1(3/2,5/2;3;-4p^2)} \right]^2. \quad (7)$$

The dependence upon the parameters $\epsilon_{1/2}$, k_F, and ξ is therefore shown. Analytic transformations can be performed, for example, by using the reduction formulae for contiguous hypergeometric functions [19]. The numerical evaluations of appendix I of [1] are unnecessary. Using the BCS relation $\xi = 0.32\hbar v_F/\Delta_0$ and $\epsilon_{1/2} \simeq \Delta_0^2/2\epsilon_F$ gives the estimate of the prefactor $m_0^* \simeq \hbar^2 k_F^2/4\epsilon_{1/2} \simeq (5/2)(k_F\xi)^2 m_e$, where m_e is the electronic mass. For $p = 4.47$ this gives the estimated size of the bare core mass $m_0^* \simeq 50 m_e$.

2. Core dielectric constant

Due to the discrete energy levels in the vortex core, there is a screening of the vortex mass. Then, the mass is given by $m^* = m_0^*/(1 + 2\pi\, e^2 M(0))$ and it is shown here that progress can be made in analytically evaluating the factor $M(0)$. Physically, $M(0)$ is a polarizability due to the core quasi-particles. The core dielectric constant $\epsilon_{\text{core}} = 1 + 2\pi\, e^2 M(0)$ can be written as [1] $\epsilon_{\text{core}} = 1 + (e^2/\xi) L_1/\epsilon_{+-} L_2^2$ where the excitation energy $\epsilon_{+-} = 2\epsilon_{1/2}$. The integral L_2 is the square of the normalization integral and has been evaluated above in the denominator of g_x,

$$L_2 = \int_0^\infty x\,\mathrm{d}x\, e^{-x} [J_0^2(px) + J_1^2(px)] = \frac{2}{\pi} \frac{\boldsymbol{E}(-4p^2)}{(1+4p^2)} + \frac{3}{2} p^2 {}_2F_1(3/2,5/2;3;-4p^2). \quad (8)$$

The other integral

$$L_1 = 4 \int_0^\infty \mathrm{d}x \frac{\sinh(xd/\xi)}{\cosh(xd/\xi) - 1} f^2(x) \quad (9)$$

has

$$f(x) = \int_0^\infty y\,\mathrm{d}y\, J_0(py) J_1(py) J_1(xy)\, e^{-y}. \quad (10)$$

By way of the expansion of the product of two Bessel functions of like argument [19]

$$J_0(z)J_1(z) = \sum_{m=0}^\infty \frac{(-1)^m (1/2)^{2m+1} (m+2)_m}{(m+1)(m!)^3} z^{2m+1} \quad (11)$$

where $(m+2)_m = \Gamma(2m+2)/\Gamma(m+2)$, it is possible to write $f(x)$ as an infinite series. (By the duplication formula, one can write $(m+2)_m = (2\pi)^{-1/2} 2^{2m+7/2} \Gamma(m+5/2)/(2m+3)(2m+2)$.) Two equivalent forms of the result are

$$f(x) = \frac{x}{4} \sum_{m=0}^\infty \frac{(-1)^m}{4^m} \frac{(m+2)_m (2m+3)!}{(m+1)(m!)^3} (k_F\xi)^{2m+1} {}_2F_1(m+2, m+5/2; 2; -x^2) \quad (12)$$

$$f(x) = \sum_{m=0}^\infty \frac{(-1)^m}{2^{2m+1}} \frac{(m+2)_m (2m+3)!}{(m+1)(m!)^3} (k_F\xi)^{2m+1} (1+x^2)^{-(m+3/2)} P_{2(m+1)}^{-1}(1/\sqrt{1+x^2}) \quad (13)$$

as

$$F(a, a+1/2; 2; z) = 2(-z)^{-1/2}(1-z)^{1/2-a} P_{2a-2}^{-1}[(1-z)^{-1/2}] \quad (14)$$

for $-\infty < z < 0$. In equations (13) and (14), P_m^n is the associated Legendre function of the first kind [19].

Before briefly discussing the nature of the integrand of L_1, one can note that many general forms of this integral can be written, for example

$$L_1 = 4\int_0^\infty \coth(xd/2\xi) f^2(x)\,\mathrm{d}x \tag{15}$$

and several others based upon integration by parts. By using the expansion $J_1(xy) = xy/2 - x^3y^3/16 + \mathrm{O}(x^5)$, one can directly examine the small-argument behaviour of $f(x)$

$$f(x \ll 1) = \frac{x}{2} 3k_\mathrm{F}\xi[{}_2F_1(3/2, 5/2; 2; -4k_\mathrm{F}^2\xi^2) - (5/2)x^2{}_2F_1(3/2, 7/2; 2; -4k_\mathrm{F}^2\xi^2) \\ +(105/8)k_\mathrm{F}^2\xi^2x^2{}_2F_1(5/2, 9/2; 3; -4k_\mathrm{F}^2\xi^2)]. \tag{16}$$

This limiting behaviour of f is sufficient to guarantee the convergence of L_1 near the origin.

Extending the expansion (16), it is possible to write $f(x)$ as a power series in odd powers of x, $f(x) = \sum_{k=0}^\infty c_k x^{2k+1}$, where the coefficients are given by

$$c_k = p\frac{(-1)^k}{2^{2(k+1)}}\frac{(2k+5)!}{k!(k+1)!}{}_3F_2(3/2, k+2, k+5/2; 2, 2; -4p^2). \tag{17}$$

This series is developed by writing the series expansion of $J_1(xy)$ in equation (10) and using the integral (Laplace transform)

$$\int_0^\infty y^a J_0(y)J_1(y)\,\mathrm{e}^{-by}\,\mathrm{d}y = \frac{1}{2b^{a+2}}\Gamma(a+2){}_3F_2(3/2, 1+a/2, (a+3)/2; 2, 2; -4/b^2). \tag{18}$$

Although the coefficients c_k involve the generalized hypergeometric function ${}_3F_2$, there is a very useful special relation between the numerator and denominator parameters. Specifically, the numerator parameter $k+2$ exceeds the denominator parameter 2 by the non-negative integer k. Therefore, for $k = 0$ this ${}_3F_2$ immediately reduces to the Gauss hypergeometric function and for $k \geqslant 1$ the ${}_3F_2$ can be written as a sum of $(k+1)$ ${}_2F_1$ [20]. As an illustration, the $k = 0$ and $k = 1$ cases are particularly simple,

$$c_0 = 30p\,{}_2F_1(3/2, 5/2; 2; -4p^2)$$

$$c_1 = -\frac{315}{2}p\,{}_3F_2(3/2, 3, 7/2; 2, 2; -4p^2) \\ = -\frac{315}{2}p\left[{}_2F_1(3/2, 7/2; 2; -4p^2) - \frac{21}{4}p^2{}_2F_1(5/2, 9/2; 3; -4p^2)\right]. \tag{19}$$

For $p = 4.47$, these evaluate to $c_0 \simeq 0.082\,2294$ and $c_1 \simeq -0.041\,3491$. The general reduction is given by

$${}_3F_2(b, c, a+k; d, a; x) = \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(b)_\ell(c)_\ell}{(a)_\ell(d)_\ell}x^\ell{}_2F_1(b+\ell, c+\ell; d+\ell; x) \tag{20a}$$

where $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol. As applied to the ${}_3F_2$ factor in equation (17), this gives

$${}_3F_2(3/2, k+5/2, k+2; 2, 2; -4p^2) = \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(3/2)_\ell(k+5/2)_\ell}{[(\ell+1)!]^2} \\ \times(-1)^\ell(2p)^{2\ell}{}_2F_1(3/2+\ell, k+5/2+\ell; \ell+2; -4p^2). \tag{20b}$$

By taking into account the sharp peak in the coth function at the origin and using equation (15), it is possible to write for L_1 the analytic approximation

$$\frac{L_1}{4} \simeq \frac{2\xi}{d}\int_0^{2\xi/d}\frac{f^2(x)}{x}\,\mathrm{d}x + \int_{2\xi/d}^\infty f^2(x)\,\mathrm{d}x. \tag{21}$$

That is, both the large- and small-argument asymptotic forms of coth have been used. This approximation is illustrated in the appendix for a simple class of smooth functions.

It is possible to write exactly that

$$f^2(x) = x^2 \sum_{j=0}^{\infty} \sum_{k=0}^{j} c_{j-k} c_k x^{2j}. \tag{22}$$

It is not possible to insert this form into equation (15) and interchange the summation and integration because the integrand does not appear to be uniformly convergent. However, it may be possible to use the approximation (21) in some form. If there is a sufficient decrease in f^2 for large argument, then the second term of equation (21) may be ignored. This leads to the rough approximation

$$L_1 \approx 2 \left(\frac{2\xi}{d}\right)^3 \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{c_{j-k} c_k}{(j+1)} \left(\frac{2\xi}{d}\right)^{2j}. \tag{23}$$

Although the coefficients c_k appear to alternate in sign, the prefactor $1/(j+1)$ in equation (17) is not rapidly decreasing, and it therefore appears that many terms should be taken in this approximation.

3. Alternative evaluation of the normalization integral

There are several other possibilities for analytically evaluating the square of the normalization constant, L_2. There being a close relation between complete elliptic integrals and Legendre functions, either of these sets of functions can be used instead of the method above. The use of Legendre functions Q_ν of the second kind is illustrated here, providing an alternative to employing the Gauss hypergeometric function F as previously mentioned.

A starting point can be the two Laplace transforms [19]

$$\int_0^\infty \mathrm{e}^{-\alpha x} J^2_{\substack{1\\0}}(\beta x)\,\mathrm{d}x = \frac{1}{\pi\beta} Q_{\pm 1/2}\left(\frac{\alpha^2 + 2\beta^2}{2\beta^2}\right) \qquad \mathrm{Re}\,\alpha > 0. \tag{24}$$

Then differentiation with respect to α and the recursion relations [19]

$$\begin{aligned} Q'_{-1/2}(z) &= \frac{1}{2(z^2-1)}[Q_{1/2}(z) - zQ_{-1/2}(z)] \\ Q'_{1/2}(z) &= \frac{3}{2(z^2-1)}[Q_{3/2}(z) - zQ_{1/2}(z)] \end{aligned} \tag{25}$$

gives the results

$$\begin{aligned} \int_0^\infty x\,\mathrm{e}^{-\alpha x} J_0^2(\beta x)\,\mathrm{d}x &= -\frac{2\beta}{\pi\alpha}\frac{1}{(\alpha^2+4\beta^2)} \\ &\quad\times \left[Q_{1/2}\left(\frac{\alpha^2+\beta^2}{2\beta^2}\right) - \left(\frac{\alpha^2+\beta^2}{2\beta^2}\right) Q_{-1/2}\left(\frac{\alpha^2+\beta^2}{2\beta^2}\right)\right] \\ \int_0^\infty x\,\mathrm{e}^{-\alpha x} J_1^2(\beta x)\,\mathrm{d}x &= -\frac{6\beta}{\pi\alpha}\frac{1}{(\alpha^2+4\beta^2)} \\ &\quad\times \left[Q_{3/2}\left(\frac{\alpha^2+\beta^2}{2\beta^2}\right) - \left(\frac{\alpha^2+\beta^2}{2\beta^2}\right) Q_{1/2}\left(\frac{\alpha^2+\beta^2}{2\beta^2}\right)\right]. \end{aligned} \tag{26}$$

Putting $\beta = 1$ and $\alpha = 1/k_F\xi$, making a change of variable in the integral of equation (8), and combining the expressions of equation (26) gives for the square of the normalization constant

$$L_2(k_F\xi) = -\frac{2}{\pi}\frac{(k_F\xi)^{-1}}{[(k_F\xi)^{-2}+4]}[3Q_{3/2}(z_F) + (1-3z_F)Q_{1/2}(z_F) - z_F Q_{-1/2}(z_F)] \qquad (27)$$

where $z_F \equiv 1 + 1/2(k_F\xi)^2$. When $z_F = 1.025$, this evaluates to $L_2 \simeq 0.140$, in agreement with equation (8).

4. Summary

The vast majority of numerical computations in [1] have been shown to be unnecessary. Given that one has made a choice to work analytically with approximate wavefunction solutions of the Bogoliubov–de Gennes equations, one would like to remain within this framework. Alternatively, an entirely numerical approach may be more suitable.

The approximate $f^{\pm}_{1/2}$ wavefunctions used here do vary over both the Fermi wavelength $\lambda_F = 2\pi/k_F$ and coherence length ξ. However, further study may be required to show that they adequately reflect the variation of the pair potential over these two length scales at low temperature. The self-consistent solutions of [4] show quantitatively the pair potential at various temperatures.

The analytic results developed here are needed to further the analysis of the vortex mass and dynamic mobility, even for a superconductor with an s-wave and BCS-like gap. The detailed microscopic parameter dependences have been displayed, and this has been done fully for the unscreened or bare core contribution. Equation (7) shows that the bare mass is relatively insensitive to the product $k_F\xi$ over a range of values appropriate to the high-T_c cuprates. The prefactor $\hbar^2 k_F^2/4\epsilon_{1/2} \simeq (5/2)(k_F\xi)^2 m_e$ directly gives the order of magnitude of the unscreened vortex mass.

In the absence of pinning, the vortex mobility is $\tilde{\mu}_v(\omega, B, T) = 1/(\eta - i\omega\mu)$, where η is the drag coefficient and μ the mass per unit length. In the further absence of the viscous force, the mobility reduces to $\tilde{\mu}_v = i/\omega\mu$, as expected. This latter situation may approximately hold at extremely low temperatures for ultraclean type-II superconductors. As noted in this paper, ultraclean superconductors can have drastically altered vortex dynamics.

The normalization integral of the approximate wavefunctions $f^{\pm}_{1/2}$ for the last occupied and first unoccupied bound core states has been evaluated in several different but equivalent ways, including equations (8) and (27). These are valuable to have in calculating matrix elements, especially observables.

The function $f(x)$ of equation (10) is related to the Fourier transform of an integrated charge density-current correlation function. This function has been evaluated in a variety of methods, including as a power series in $k_F\xi$, equations (12)–(13), and as a power series in x, equation (17). It can be noted that the dipolar charge distribution generated by rectilinear vortex motion has been anticipated for anisotropic superconductors in [12]. [1] does not cite the basic early work of Bardeen and Stephen [21].

Although the coefficients c_k in the power series expansion (17) of $f(x)$ contain the generalized hypergeometric function ${}_3F_2$, these have a special form as noted so that they reduce to sums of the Gauss hypergeometric function $F(a, b; c; z)$. (Additional information on the special functions employed here can be found in [20].)

The only quantity remaining to be evaluated exactly in the core polarizability $M(0)$ is the integral L_1, equation (9) or (15). An approximation to this integral has been presented in (21), with a crude approximation explicitly involving the c_k given in (23). Further study should elucidate the limits to these approximations. Alternative approaches include of course the use of (12) or (13) in equation (15). This would allow the analytic characterization of the Coulomb screening as a function of interlayer spacing, Fermi wavenumber, and coherence length.

The results presented here for the vortex mass apply at absolute zero. Their quantitative form makes them suitable for comparison with other methods, or for contrast with results at non-zero temperature. By knowing the microscopic parameter dependences, the differences for non-s-wave superconductors can be better understood. For vortices with an altered bound core state structure, some modification of the inertial mass can be expected.

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Appendix. The approximation (21)

This appendix illustrates the analytic approximation, equation (21) to equation (15). Some simple functions are taken in the integrand and the numerical values of the results are used to verify the approximation. Of course the class of functions for f^2 in (15) must be restricted in some way. A very tight characterization of the function multiplying coth in (15) is taken here, namely that it be smooth, meaning infinitely differentiable. It is also assumed to be sufficiently decreasing for large argument that the integral (15) is convergent. Similarly, for small x, it is required that the multiplying function of coth behave as x^q for $q > 0$ in order to ensure convergence at the lower limit.

A simple class of functions satisfying the above requirements is $x^p\,\mathrm{e}^{-x}$, where $p > 0$. Here the values of

$$I_p \equiv \int_0^\infty x^p\,\mathrm{e}^{-x}\coth x\,\mathrm{d}x \tag{A.1}$$

are compared to the values of the approximation

$$A_p \equiv \int_0^1 x^{p-1}\,\mathrm{e}^{-x}\,\mathrm{d}x + \int_1^\infty x^p\,\mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(p+1) + \Gamma(p,0,1) - \Gamma(p+1,0,1) \tag{A.2}$$

where the generalized incomplete gamma function $\Gamma(a, z_1, z_2) = \Gamma(a, z_1) - \Gamma(a, z_2)$ in terms of the incomplete gamma function $\Gamma(a, z)$.

For $p = 1/4$, one has $I_{1/4} \simeq 3.921\,35$, while $A_{1/4} = \Gamma(1/4) + \Gamma(5/4) - \Gamma(1/4, 1) - \Gamma(5/4, 0, 1) \simeq 3.8088$, there being a relative error of about 2.9%. For $p = 1/3$, one has $I_{1/3} \simeq 2.985\,95$, while $A_{1/3} = \Gamma(1/3)+\Gamma(4/3)-\Gamma(1/3, 1)-\Gamma(4/3, 0, 1) \simeq 2.875\,88$, with a relative error of 3.7%. For $p = 1$, $I_1 = \pi^2/4 - 1 \simeq 1.4674$, $A_1 = 1 + 1/\mathrm{e} \simeq 1.367\,88$, with a 6.8% relative error. When $p = 2$, $I_2 = (7/2)\zeta(3) - 2 \simeq 2.2072$, where ζ is the Riemann zeta function. The approximation (A.2) gives $A_2 = 1 + 3/e \simeq 2.103\,64$, with a 4.7% error. When $p = 3$, $I_3 = \pi^4/8 - 6 \simeq 6.176\,14$, $A_3 = 2 + 11/e \simeq 6.046\,67$, with a 2.1% error. For $p = 4$, one has $I_4 \simeq 24.2171$, and $A_4 = 6 + 49/e \simeq 24.0261$ with less than one percent error.

Another set of examples could be based upon the integral

$$I_p = \int_0^\infty x^p J_1(x)\coth x\,\mathrm{d}x \qquad p > -1 \tag{A.3}$$

for which an approximation is

$$A_p = \int_0^1 x^{p-1} J_1(x)\,\mathrm{d}x + \int_1^\infty x^p J_1(x)\,\mathrm{d}x$$
$$= 2^p \frac{\Gamma(p/2+1)}{\Gamma(1-p/2)} - {}_1F_2(1+p/2; 2, 2+p/2; -1/4)/2(2+p)$$
$$+ {}_1F_2((1+p)/2; 2, (p+3)/2; -1/4)/2(1+p). \qquad \text{(A.4)}$$

Here $A_{-1/2} = \Gamma(3/4)/\sqrt{2}\Gamma(5/4) + {}_1F_2(1/4; 5/4, 2; -1/4) - {}_1F_2(3/4; 7/4, 2; -1/4)/3 \simeq 1.615\,61$, $A_{1/2} = \sqrt{2}\Gamma(5/4)/\Gamma(3/4) + {}_1F_2(3/4; 7/4, 2; -1/4)/3 - {}_1F_2(5/4; 2, 9/4, -1/4)/5 \simeq 1.175\,49$, $A_0 = J_0(1) + {}_1F_2(1/2; 2, 3/2; -1/4)/2 \simeq 1.244\,88$ $A_1 = 2 - J_0(1) - {}_1F_2(3/2; 2, 5/2; -1/4)/6 \simeq 1.080\,27$, $A_2 = -J_2(1) + {}_1F_2(3/2; 2, 5/2; -1/4)/6 \simeq 0.039\,629\,2$, $A_3 = -3 + J_2(1) - {}_1F_2(5/2; 2, 7/2; -1/4)/10 \simeq -2.976\,45$, $A_4 = -4J_3(1) + J_4(1) + {}_1F_2(5/2; 2, 7/2; -1/4)/10 \simeq 0.015\,579\,1$, and $A_5 = 45 + 4J_3(1) - J_4(1) - {}_1F_2(7/2; 2, 9/2; -1/4)/14 \simeq 45.0111$. The appearance of $J_\alpha(1)$ in A_p follows from the reduction of ${}_1F_2$ and the relation $J_\alpha(1) = {}_0F_1(--; \alpha+1; -1/4)/2^\alpha\Gamma(\alpha+1)$.

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